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Abstract

Let δ be a derivation on R . A ring R is called δ -quasi-Baer (resp. quasi-Baer) if the right annihilator of every δ -ideal (resp. ideal) of R is generated by an idempotent of R . In this note first we give a positive answer to the question posed in Han et al. [7], then we show that R is δ -quasi-Baer iff the differential polynomial ring $S = R[x; \delta]$ is quasi-Baer iff S is δ -quasi-Baer for every extended derivation δ on S of δ . This results is a generalization of Han et al. [7], to the case where R is not assumed to be δ -semiprime.

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ABSTRACT. Let δ be a derivation on R . A ring R is called δ -quasi-Baer (resp. quasi-Baer) if the right annihilator of every δ -ideal (resp. ideal) of R is generated by an idempotent of R . In this note first we give a positive answer to the question posed in Han et al. [7], then we show that R is δ -quasi-Baer iff the differential polynomial ring $S = R[x; \delta]$ is quasi-Baer iff S is $\bar{\delta}$ -quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ . This results is a generalization of Han et al. [7], to the case where R is not assumed to be δ -semiprime.

Throughout this note R denotes an associative ring with unity, $\delta : R \rightarrow R$ is derivation of R , that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x; \delta]$ the skew polynomial ring whose elements are the polynomials $\sum_{i=0}^n r_i x^i \in R$, $r_i \in R$, where the addition is defined as usual and the multiplication by $xb = bx + \delta(b)$ for any $b \in R$. For a nonempty subset X of a ring R , we write $r_R(X) = \{c \in R \mid dc = 0 \text{ for any } d \in X\}$ which is called the *right annihilator* of X in R .

Recall from [9] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [9] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$ -regular rings. The class of Baer rings includes the von Neumann algebras. In [6] Clark defines a ring to be *quasi-Baer* if the right annihilator of every ideal is generated, as a right ideal, by an idempotent. Moreover, he shows the left-right symmetry of this condition by proving that R is quasi-Baer if and only if the left annihilator of every left ideal is generated, as a left ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Further work on quasi-Baer rings appears in [3, 4, 5, 10, 11]. An ideal I of R is called δ -ideal if $\delta(I) \subseteq I$. R is called δ -quasi-Baer if the right annihilator of every δ -ideal of R is generated by an idempotent of R . Clearly each quasi-Baer ring is δ -quasi-Baer. But the converse is not true (see [7] Example). R is said to be *reduced* if R has no nonzero nilpotent elements. Note that in a reduced ring R , R is Baer if and only if R is quasi-Baer.

In [1], Armendariz has shown that if R is reduced, then R is Baer if and only if the polynomial ring $R[x]$ is a Baer ring. Han et al. [7], have generalized this result by showing that if R is δ -semiprime (i.e., for any δ -ideal I of R , $I^2 = 0$ implies $I = 0$), then R is a δ -quasi-Baer ring if and only if the

Ore extension $R[x; \delta]$ is a quasi-Baer ring.

Han et al. (2000) posed this question: If $e(x) \in R[x; \delta]$ is a left semicentral idempotent, then does there exist a left semicentral idempotent $e_0 \in R$ such that $e(x)R[x; \delta] = e_0R[x; \delta]$? In this note first we give a positive answer to this question, then we show that R is δ -quasi-Baer if and only if the differential polynomial ring $S = R[x; \delta]$ is quasi-Baer if and only if S is $\bar{\delta}$ -quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ . This result is a generalization of Han et al. [7], to the case where R is not assumed to be δ -semiprime.

For a ring R with a derivation δ , there exists a derivation on $S = R[x; \delta]$ which extends δ . For example given in [7], consider an inner derivation $\bar{\delta}$ on S by x defined by $\bar{\delta}(f(x)) = xf(x) - f(x)x$ for all $f(x) \in S$. Then $\bar{\delta}(f(x)) = \delta(a_0) + \cdots + \delta(a_n)x^n$ for all $f(x) = a_0 + \cdots + a_nx^n \in S$ and $\bar{\delta}(r) = \delta(r)$ for all $r \in R$, which means that $\bar{\delta}$ is an extension of δ . We call such a derivation $\bar{\delta}$ on S an *extended derivation* of δ . For each $a \in R$ and nonnegative integer n , there exist $t_0, \dots, t_n \in \mathbb{Z}$ such that $x^na = \sum_{i=0}^n t_i \delta^{n-i}(a)x^i$.

Lemma 1. (Han et al. Lemma 1) *Let R be a ring with a derivation δ and $\bar{\delta}$ be an extended derivation of δ on $S = R[x; \delta]$. If I is a δ -ideal of R , then $I[x; \delta]$ is $\bar{\delta}$ -ideal of S .*

Proof. By ([8], Lemma 1.3), $I[x; \delta]$ is an ideal of S . Let $f(x) = a_0 + \cdots + a_nx^n \in I[x; \delta]$. For each i , $\bar{\delta}(a_ix^i) = \bar{\delta}(a_i)x^i + a_i\bar{\delta}(x^i) = \delta(a_i)x^i + a_i\delta(x^i) \in I[x; \delta]$. Hence $I[x; \delta]$ is a $\bar{\delta}$ -ideal of S . \square

Now we give a positive answer to the question posed in Han et al. [7].

Theorem 2. *Let I be a δ -ideal of R and $S = R[x; \delta]$. If $r_S(I[x; \delta]) = e(x)S$ for some idempotent $e(x) = e_0 + e_1x + \cdots + e_nx^n \in S$, then $r_S(I[x; \delta]) = e_0S$.*

Proof. Since $Ie(x) = 0$, we have $Ie_i = 0$ for each $i = 0, \dots, n$. Hence $0 = \delta(Ie_i) = \delta(I)e_i + I\delta(e_i)$ for $i = 0, \dots, n$. Since I is δ -ideal and $Ie_i = 0$, so $I\delta(e_i) = 0$ for each $i = 0, \dots, n$. By a similar argument we can show that $I\delta^k(e_i) = 0$ for each $i = 0, \dots, n$ and $k \geq 0$. Hence $\delta^k(e_i) \in r_S(I[x; \delta])$ for each $i = 0, \dots, n$ and $k \geq 0$. Thus $\delta^k(e_i) = e(x)\delta^k(e_i)$ and that $e_n\delta^k(e_i) = 0$ for each $i = 0, \dots, n$ and $k \geq 0$. Hence $\delta^k(e_i) = (e_0 + e_1x + \cdots + e_{n-1}x^{n-1})\delta^k(e_i)$ and that $e_{n-1}\delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0$. Continuing in this way, we have $e_j\delta^k(e_i) = 0$ for each $i \geq 0, k \geq 0, j = 1, \dots, n$. Thus $\delta^k(e_i) = e_0\delta^k(e_i)$ for each $i \geq 0, k \geq 0$. Therefore $e(x) = e_0e(x)$ and that $r_S(I[x; \delta]) = e(x)S \subseteq e_0S$. Since $\delta^k(e_0) \in r_R(I)$, so $e_0 \in r_S(I[x; \delta])$ and that $e_0S \subseteq r_S(I[x; \delta])$. Therefore $r_S(I[x; \delta]) = e_0S$. \square

Proposition 3. *Let R be a δ -quasi-Baer ring. Then $S = R[x; \delta]$ is a quasi-Baer ring.*

Proof. Let J be an arbitrary ideal of S . Consider the set J_0 of leading coefficients of polynomials in J . Then J_0 is a δ -ideal of R . Since R is δ -quasi-Baer, $r_R(J_0) = eR$ for some idempotent $e \in R$. Since $J_0e = 0$ and J_0 is δ -ideal of R , we have $J_0\delta^k(e) = 0$ for each $k \geq 0$. Hence $\delta^k(e) = e\delta^k(e)$ and $eS \subseteq r_S(J_0[x; \delta])$. Clearly $r_S(J_0[x; \delta]) \subseteq eS$. Thus $r_S(J_0[x; \delta]) = eS$. We claim that $r_S(J) = eS$. Let $f(x) = a_0 + \cdots + a_n x^n \in J$. Then $a_n \in J_0$ and that $a_n \delta^k(e) = 0$ for each $k \geq 0$. Hence $f(x)e = (a_0 + \cdots + a_{n-1}x^{n-1})e = \cdots + a_{n-1}ex^{n-1}$. Thus $a_{n-1}e \in J_0$, and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$ for each $k \geq 0$. Hence $a_{n-1}x^{n-1}e = 0$. Continuing in this way, we can show that $a_i x^i e = 0$, for each $i = 0, \dots, n$. Hence $f(x)e = 0$ and so $eS \subseteq r_S(J)$. Now, let $g(x) = b_0 + \cdots + b_m x^m \in r_S(J)$ and $f(x) = a_0 + \cdots + a_n x^n \in J$. First, we will show that $a_i x^i b_j x^j = 0$, for $i = 0, \dots, n, j = 0, \dots, m$. Since $f(x)g(x) = 0$, we have $a_n b_m = 0$. Hence $b_m \in r_R(J_0)$. Since J_0 is δ -ideal of R , $\delta^k(b_m) \in J_0$ for each $k \geq 0$ and that $b_m \in r_S(J_0[x; \delta])$. Thus $b_m = eb_m$ and $a_n x^n b_m x^m = 0$. Since $f(x)e = (a_0 + \cdots + a_n x^n)e = (a_0 + \cdots + a_{n-1}x^{n-1})e$, we have $a_{n-1}e \in J_0$ and $a_{n-1}\delta^k(e) = a_{n-1}e\delta^k(e) = 0$, for each $k \geq 0$. There exist $t_0, \dots, t_{n-1} \in \mathbb{Z}$ such that, $a_{n-1}x^{n-1}b_m x^m = a_{n-1}x^{n-1}eb_m x^m = a_{n-1}(\sum_{j=0}^{n-1} t_j \delta^{n-1-j}(e)x^j)b_m x^m = (\sum_{j=0}^{n-1} t_j a_{n-1}\delta^{n-1-j}(e)x^j)b_m x^m$. Hence $a_{n-1}x^{n-1}b_m x^m = 0$. Continuing in this way, we have $a_i x^i b_j x^j = 0$ for each i, j . Therefore $b_j \in r_S(J_0[x; \delta]) = eS$, for each $j \geq 0$. Consequently, $g(x) = eg(x)$ and $r_S(J) = eS$. Therefore S is a quasi-Baer ring. \square

Theorem 4. Let R be a ring and $S = R[x; \delta]$. Then the following are equivalent:

- (1) R is δ -quasi-Baer;
- (2) S is quasi-Baer;
- (3) S is $\bar{\delta}$ -quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ .

Proof. (1) \Rightarrow (2). It follows from Proposition 3.

(2) \Rightarrow (3). It is clear.

(3) \Rightarrow (1). Suppose that R is $\bar{\delta}$ -quasi-Baer for every extended derivation $\bar{\delta}$ on S of δ . Let I be any δ -ideal of R . Then by Lemma 1, $I[x; \delta]$ is $\bar{\delta}$ -ideal of S . Since S is $\bar{\delta}$ -quasi-Baer, $r_S(I[x; \delta]) = e(x)S$ for some idempotent $e(x) \in S$. Hence $r_S(I[x; \delta]) = e_0 S$ for some idempotent $e_0 \in R$, by Theorem 2. Since $r_R(I) = r_S(I[x; \delta]) \cap R = e_0 R$, R is δ -quasi-Baer. \square

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